Half-quantum groups at roots of unity, path algebras and representation type

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Abstract

Let $\mathcal{G}$ be a simple Lie algebra of type **A**, **D** or **E** and q a primitive root of unity of order $n \geq 5$. We show that the finite dimensional half-quantum group $\mathsf{u}_q^+(\mathcal{G})$ is of wild representation type, except for $\mathcal{G} = sl_2$. Moreover, the algebra $\mathsf{u}_q^+(\mathcal{G})$ is an admissible quotient of the path algebra of the Cayley graph of the abelian group $(\mathbf{Z}/n\mathbf{Z})^t$ with respect to the columns of the $t \times t$ Cartan matrix of $\mathcal{G}$.

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1 Introduction

Let $\mathcal{G}$ be a simple Lie algebra corresponding to a symmetric positive definite Cartan matrix of type **A**, **D** or **E**. We consider in this paper the finite dimensional Hopf algebras $\mathsf{u}_q^+(\mathcal{G})$ (see ([28, 29, 30, 33, 34]) for q a root of unity; they are quotients of the upper triangular sub-algebra of the quantum groups $U_q(\mathcal{G})$ introduced by Drinfel'd and Jimbo [13, 14, 22, 23]. Those structures gives rise to solutions of the Yang-Baxter equation through the modular category of representations and the universal R-matrix acting on it. They have been useful in knot theory and 3-manifold invariants, see for instance [4, 25, 42].

The representation theory of quantum groups at roots of unity is not fully understood, see [42, 7, 8, 40]. Actually we will show that $\mathsf{u}_q^+(\mathcal{G})$ is of wild representation type except for $\mathcal{G} = sl_2$. It is known that finite dimensional algebras are either of finite, tame or wild representation type, see [15, 11, 20]; an algebra is said to be tame if the isomorphism classes of indecomposable modules of a fixed dimension are almost all in a finite number of 1-parameter algebraic families. Tame algebras are suitable for reaching a classification of indecomposable modules, by contrast the module

category of a wild representation type algebra contains the representation theory of any other finitely generated algebra, see [17]. For an account of the theory, see for instance [15, 32] or [27]. The sl_2-case corresponds to a finite representation type quantum group, see [5]. Suter and Xiao independently ([41, 43]), have computed the Morita reduction of the entire $\mathsf{u}_q(sl_2)$, obtaining that this algebra is of tame representation type, and providing canonical forms for the indecomposable modules. Notice that the Morita reduction of $\mathsf{u}_q(sl_2)$ is not anymore a Hopf algebra, hence the monoidal structure on the canonical forms is unknown. It is very likely that the entire finite dimensional quantum groups $\mathsf{u}_q(\mathcal{G})$ are of wild representation type if $\mathcal{G}$ is different from sl_2.

A starting point for our considerations is an easy observation concerning simple $\mathsf{u}_q^+(\mathcal{G})$-modules: all of them are one-dimensional. This implies that $\mathsf{u}_q^+(\mathcal{G})$ can be presented as a quotient of the path algebra of its canonical quiver $Q_{\mathcal{G}}$ by an ideal of relations J contained in the square of the ideal generated by the arrows of the quiver and containing some power of it, such a presentation is called admissible, see [16, 18, 19]. It insures the unicity of $Q_{\mathcal{G}}$; in turn, simple module of an admissible quotient of a path algebra are one-dimensional and in one-to-one correspondence with the vertices of the quiver. Moreover, the dimension of Ext^1 between two simple modules is the number of arrows between the corresponding vertices.

We describe now the quiver $Q_{\mathcal{G}}$, which is a finite oriented graph: let t be the size of the Cartan matrix C of $\mathcal{G}$ (t is the number of simple roots of $\mathcal{G}$) and let q be a root of unity of order n. The set of vertices of $Q_{\mathcal{G}}$ is the abelian group $G = < K_1, \ldots, K_t \mid K_i^n = 1, \ K_i K_j = K_j K_i >$. The quiver $Q_{\mathcal{G}}$ is the Cayley graph of G with respect to the set of elements given by the columns of C, in other words each vertex $K^x = K_1^{x_1} K_2^{x_2} \ldots K_n^{x_n}$ is the target of t arrows coming from $\{K^{x-a_{\text{-},j}}\}_{j=1,\ldots,t}$ where $(a_{\text{-},j})_{j=1,\ldots,t}$ are the columns of C and $K^{x+a_{\text{-},j}}$ is the element $K_1^{x_1+a_{1,j}} K_2^{x_2+a_{2,j}} \ldots K_t^{x_t+a_{t,j}}$ of G. In case of sl_2 the Cartan matrix is $C = (2)$, the quiver Q_{sl_2} has n vertices $\{1, K, \ldots, K^{n-1}\}$ with arrows $K^{x-2} \to K^x$; this quiver is connected for n odd and have two connected components for n even. Its quantum structure and monoidal category of representations has been described in [5], notice that there are only a finite number of isomorphism classes of indecomposable $\mathsf{u}_q^+(sl_2)$-modules and that their tensor product is given through Clebsch-Gordan-like formulas.

The Cayley graph of a group is usually builded using a set of generators of the group. In our case the columns of C generates G if and only if the

determinant of C is invertible modulo n. The quiver $Q_{\mathcal{G}}$ is connected exactly in this case.

The isomorphism $\chi_q : \mathsf{u}_q^+(\mathcal{G}) \longrightarrow kQ_{\mathcal{G}}/J$ is given explicitely. The formulas are provided by structure results about Hopf bimodules obtained in [6] and by the quantum Fourier transform. The latter extends to the level of Hopf bimodules the usual Fourier transform $\chi : kG \longrightarrow k^G$, i.e. the linear extension of the group isomorphism between a finite abelian group G and its character group obtained through a choice of generators and roots of unity of the field k. Notice that χ is actually a Hopf algebra isomorphism as well as χ_q for a natural Hopf structure on $kQ_{\mathcal{G}}/J$ that we provide. The way χ_q is obtained guarantees that the formulas gives a well defined isomorphism of Hopf algebras. Alternatively, it is possible to ignore the origin of the formulas and to verify directly that they provides an isomorphism of associative algebras. With this point of view, the coalgebra structure on the admissible path algebra quotient is obtained from $\mathsf{u}_q^+(\mathcal{G})$ by structure transport through χ_q.

The presentation of $\mathsf{u}_q^+(\mathcal{G})$ given by the model of paths with comultiplication of $Q_{\mathcal{G}}$ is related but different from the canonical bases of Lusztig ([30]). The purpose of this paper is to link quantum groups at roots of unity with results concerning finite dimensional algebras. C.M. Ringel has obtained another approach to $\mathsf{u}_q^+(\mathcal{G})$, by means of indecomposable representations of the quiver given by the Dynkin diagram associated to C over a finite field, equipped with a Hall algebra like multiplication, see [35, 21].

The representation type of $\mathsf{u}_q^+(\mathcal{G})$ is obtained in the last section either through the path presentation and 2-nilpotent quotients, or by an induction procedure from a sub-algebra.

We also use the quiver presentation in order to show that although $\mathsf{u}_q^+(\mathcal{G})$ is not quasitriangular, there is an algebra automorphism of a twisted tensor product $\mathsf{u}_q^+(\mathcal{G}) \otimes_q \mathsf{u}_q^+(\mathcal{G})$ transforming the comultiplication to its opposite.

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2 Quantum paths for half quantum groups

A half-quantum group is a finite dimensional k-Hopf algebra $\mathsf{u}_q^+(\mathcal{G})$ associated to a symmetric Cartan matrix C and a root of unity, the definition is given below. This terminology (see [10]) has been introduced since the "usual" quantum groups $\mathsf{u}_q(\mathcal{G})$ of Drinfel'd and Jimbo are easy quotients

of the Drinfel'd double of $\mathsf{u}_q^+(\mathcal{G})$. Half-quantum groups at roots of unity and their representation theory have direct use in topological quantum field theories, see for instance [10, 26].

A symmetric matrix $C = (a_{i,j})$ of size $t \times t$ is a Cartan matrix if $a_{i,i} = 2$ for $i = 1, \ldots, t$ and $a_{i,j} \in \{0, -1\}$ for $i \neq j$. Symmetric $t \times t$ Cartan matrices and graphs with no loops and no multiple edges on t vertices are in natural bijection. Positive definite Cartan matrices corresponds to graphs of type $\mathbf{A}$, $\mathbf{D}$, or $\mathbf{E}$, see for instance [2]. A positive definite symmetric Cartan matrix C provides a semi-simple Lie algebra $\mathcal{G}$ with all roots of the same length.

In order to define the corresponding half-quantum group, let q be a root of unity of order $n \geq 5$ in the ground field. Set $e = n$ if n is odd and $e = \frac{n}{2}$ if n is even. The associative algebra $\mathsf{u}_q^+(\mathcal{G})$ has generators $K_1, \ldots, K_t, E_1, \ldots, E_t$ subjected to relations that we split in two sorts:

- **I**
$$K_i^n = 1, \ K_iK_j = K_jK_i, \ K_iE_jK_i^{-1} = q^{a_{i,j}}E_j$$

- **II**
$$E_i^e = 0, \ E_iE_j = E_jE_i \text{ if } a_{i,j} = 0,$$
$$E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \text{ if } a_{i,j} = -1.$$

The comultiplication $\Delta : \mathsf{u}_q^+(\mathcal{G}) \longrightarrow \mathsf{u}_q^+(\mathcal{G}) \otimes \mathsf{u}_q^+(\mathcal{G})$ is a coassociative morphism of algebras defined on the generators:
$$\Delta K_i = K_i \otimes K_i \quad \Delta E_i = K_i \otimes E_i + E_i \otimes 1.$$

The two-sided ideal of relations is preserved by Δ. Lusztig ([28, 29]) has proved that $\mathsf{u}_q^+(\mathcal{G})$ is finite dimensional using an action by algebra automorphisms of the braid group corresponding to C, and constructing root vectors associated to non simple roots.

Our first purpose is to give another presentation of $\mathsf{u}_q^+(\mathcal{G})$ through the path algebra of a specific quiver $Q_{\mathcal{G}}$. Notice first that $\mathsf{u}_q^+(\mathcal{G})$ contains the group algebra kG of $G = < K_1, \ldots, K_t \mid K_iK_j = K_jK_i, \ K_i^n = 1 >$ as a sub-Hopf algebra. We assume that the order of the group $|G|$ is invertible in the field k. Recall the Fourier transform which is a Hopf algebra isomorphism

$$\chi : k^G \longrightarrow kG$$

where $k^G = \{f : G \to k\}$ is the usual commutative algebra of functions on a set provided with the comultiplication given by $\Delta f(x, y) = f(xy)$ (we identify $k^G \otimes k^G$ and $k^{G \times G}$ since G is finite), while kG is the vector space of formal linear combinations of elements of G equipped with the multiplication

induced by the group law on the basis elements and $\Delta s = s \otimes s$ for each $s \in G$. In other words, kG is the dual Hopf algebra of k^G. In this setting we have

$$\chi\left(\delta_{K^c}\right) = \frac{1}{|G|} \sum_{K^x \in G} q^{-cx} K^x$$

$$\text{and} \ \ \chi^{-1}\left(K^a\right) = \sum_{K^x \in G} q^{ax} \delta_{K^x}$$

with the usual convention that $K^a = K_1^{a_1} \ldots K_t^{a_t}$ for $a = (a_1, \ldots, a_t)$ and that δ_{K^a} is the Dirac mass pointing out K^a. Moreover $q^{ax} = q^{a_1 x_1 + \ldots + a_t x_t}$ for $x = (x_1, \ldots, x_t)$.

The Fourier transform χ extends to $\mathsf{u}_q^+(\mathcal{G})$. In order to make this precise we consider an intermediate Hopf algebra $\mathsf{v}_q^+(\mathcal{G})$ given by the same set of generators subjected only to relations of type **I**. We assert that $\mathsf{v}_q^+(\mathcal{G})$ is isomorphic to the path algebra of a quiver $Q_{\mathcal{G}}$ (the definition is given below) depending only on the Cartan matrix corresponding to $\mathcal{G}$.

The source for finding $Q_{\mathcal{G}}$ and the extension χ_q of χ can be founded in [6]. Indeed the Hopf algebra $\mathsf{v}_q^+(\mathcal{G})$ is a tensor algebra over kG of a kG-Hopf bimodule B determined by $E_1, \ldots, E_n$. More precisely, B is a kG-bimodule which is also a kG-bicomodule with structure maps compatible with the bimodule structure. The associative tensor algebra $kG \oplus B \oplus B \otimes_{kG} B \oplus \ldots$ becomes a Hopf algebra which is identical to $\mathsf{v}_q^+(\mathcal{G})$. A theory of kG-Hopf bimodules can be developped ([6]) and it can be shown that the categories of kG and k^G-Hopf bimodules are equivalent through a Fourier transform functor. Following this track, the definitions of $Q_{\mathcal{G}}$ and χ_q are dictate. We give them directly since once the defining formulas are written, they are self-explanatory.

A quiver Q is an oriented graph given by two sets, Q_0 the set of vertices, Q_1 the set of arrows, and two maps $s, t : Q_1 \to Q_0$ providing a source and a terminus vertex to each arrow. The set k^{Q_1} of functions over the arrows has a natural bimodule structure over the commutative algebra of functions k^{Q_0}: let $\lambda \in k^{Q_0}$ and $f \in k^{Q_1}$, and the actions be given by $(\lambda f)(a) = \lambda(t(a)) f(a)$ and $(f\lambda)(a) = f(a)\lambda(s(a))$. In case Q_0 and Q_1 are finite sets, the tensor algebra $k^{Q_0} \oplus k^{Q_1} \oplus k^{Q_1} \otimes k^{Q_1} \oplus \ldots$ has a natural basis given by Dirac masses on the paths of the quiver, where a path $\alpha_n \alpha_{n-1} \ldots \alpha_1$ is a sequence of arrows such that $t(\alpha_i) = s(\alpha_{i+1})$ for each i. We denote this associative algebra k^Q, it is called the path algebra of Q. As a vector space it coincides with the finite support maps on the set of paths of the quiver. In this setting

commutative algebras corresponds to empty sets of arrows, i.e. by algebras of maps on sets (of vertices).

Definition: Let C be a $t \times t$ symmetric Cartan matrix and n a positive integer (corresponding to the order of the root of unity q). The quiver $Q_{\mathcal{G}}$ has set of vertices the group $G = <K_1, \ldots, K_t \mid K_i K_j = K_j K_i, \; K^n = 1>$. There are t arrows ending at each vertex t, which we denote $A(K^c, 1), \ldots, A(K^c, t)$. They have respectively directions $1, \ldots, t$ and sources the vertices $K^{c-a_{-,1}}, \ldots K^{c-a_{-,t}}$. In other words, $Q_{\mathcal{G}}$ is the Cayley graph of G with respect to the sub-set of G determined by the columns of C.

We quote some arithmetic of the path algebra $k^{Q_{\mathcal{G}}}$:

$$\delta_{K^d} \delta_{A(K^c, \; i)} = \delta_{A(K^c, \; i)} \quad \text{if } K^d = K^c \text{ and } 0 \text{ otherwise,}$$
$$\delta_{A(K^c, \; i)} \delta_{K^d} = \delta_{A(K^c, \; i)} \quad \text{if } K^d = K^{c-a_{i,-}} \text{ and } 0 \text{ otherwise,}$$
$$\{\delta_{K^c}\} \text{ is a complete set of primitive orthogonal idempotents.}$$

Theorem 2.1 *The associative algebra $\mathsf{v}_q^+(\mathcal{G})$ is isomorphic to the path algebra $k^{Q_{\mathcal{G}}}$.*

Proof: Let Q be the quiver $Q_{\mathcal{G}}$ associated to the Cartan matrix C. Since the path algebra is the tensor algebra over k^{Q_0} of the bimodule k^{Q_1}, an algebra map $\phi : k^Q \to A$ is uniquely and completely determined by two maps, $\phi_0 : k^{Q_0} \to A$ and $\phi_1 : k^{Q_1} \to A$ such that the first is an algebra morphism and the second is compatible with the first, which means that ϕ_1 is a k^{Q_0}-bimodule map for the bimodule structure of A induced by ϕ_0. Recall that the set of vertices of Q is the group G. In order to define $\chi_q : k^Q \longrightarrow \mathsf{v}_q^+(\mathcal{G})$ consider in degree zero the Fourier transform $[\chi_q]_0 : k^{Q_0} = k^G \longrightarrow \mathsf{v}_q^+(\mathcal{G})$

$$[\chi_q]_0 (\delta_{K^c}) = \chi(\delta_{K^c}) = \frac{1}{|G|} \sum_{K^x \in G} q^{-cx} K^x,$$

and in degree one

$$[\chi_q]_1 \left(\delta_{A(K^c, \; i)} \right) = [[\chi_q]_0 (\delta_{K^c})] \; E_i.$$

As an an immediate consequence of the commutation relation between E_i and K_j, it is interesting and easy to verify that the following equality holds in $\mathsf{v}_q^+(\mathcal{G})$; it sums up the required compatibility of $[\chi_q]_1$ with $[\chi_q]_0$:

$$[\chi_q]_0 (\delta_{K^c}) \; [\chi_q]_1 \left(\delta_{A(K^c, \; i)} \right) \; [\chi_q]_0 \left(\delta_{K^{c-a_{-,i}}} \right) = [\chi_q]_1 \left(\delta_{A(K^c, \; i)} \right).$$

Conversely, let $\Psi : \mathsf{v}_q^+(\mathcal{G}) \longrightarrow k^Q$ be defined on the generators by:

$$\Psi(K_i) = \chi^{-1}(K_i) = \sum_{K^x \in G} q^{x_i} \delta_{K^x}$$

$$\Psi(E_i) = \sum_{K^x \in G} \delta_{A(K^x, \, i)}$$

There is no difficulty for verifying that Ψ factors through the type **I** relations defining $\mathsf{v}_q^+(\mathcal{G})$, and that χ_q and Ψ are inverse one to the other.

Remark 2.2 In the presentation above of $\mathsf{v}_q^+(\mathcal{G})$ the root of unity is absent from the structure constants. Indeed, the Dirac masses on paths of Q make up a multiplicative basis of k^Q: the product of two of them is either the Dirac mass of the composite path or zero if the paths do not compose.

The structure of Hopf-bimodules over a group algebra obtained in [6] provides the path algebra k^Q with a Hopf structure. It can be recovered by structure transport through χ_q and Ψ since we have given explicit formulas. We summarize this in the following result:

Proposition 2.3 *Let q be a root of unity of order n, let C be a $t \times t$ symmetric Cartan matrix and let Q be the correponding quiver, i.e. the Cayley graph of $G = (\mathbf{Z}/n\mathbf{Z})^t$ with respect to the columns of C.*

The path algebra k^Q is a Hopf algebra isomorphic to $\mathsf{v}_q^+(\mathcal{G})$, where the comultiplication of k^Q is given by:

$$\Delta\delta_{K^c} = \sum_{K^x K^y = K^c} \delta_{K^x} \otimes \delta_{K^y}$$

$$\Delta\delta_{A(K^c, \, i)} = \sum_{K^x K^y = K^c} q^{x_i} \, \delta_{K^x} \otimes \delta_{A(K^y, \, i)} \; + \sum_{K^x K^y = K^c} \delta_{A(K^x, \, i)} \otimes \delta_{K^y}$$

Proof: It is enough to verify that the above formulas correspond through χ_q to the given ones for $\mathsf{v}_q^+(\mathcal{G})$. Consequently they provide a well defined coassociative algebra morphism $\Delta : k^Q \longrightarrow k^Q \otimes k^Q$. Observe that although k^Q is infinite dimensional, each sum in the formulas is finite since the group G is finite.

A path of length m in $Q = Q_{\mathcal{G}}$ is completely determined by its target vertex and a sequence of m directions: $A(K^c, \, i_m i_{m-1} \ldots i_1)$ is the path

ending at K^c with i_1-directional first arrow,..., and i_m-directional last arrow. In particular if each arrow of the path has same direction, we say that the path *has a direction*. We write $A(K^c, i^m)$ such an i-directional path. Recall that Dirac masses on paths of Q make up a basis for k^Q.

Theorem 2.4 *The finite-dimensional half-quantum group $\mathsf{u}_q^+(\mathcal{G})$ at a root of unity q of order n is the quotient of the path algebra $k^{Q_{\mathcal{G}}}$ by the two sided ideal J generated by*

- *Dirac masses of each path having a direction and of length e (recall that $e = n$ if n is odd and $e = \frac{n}{2}$ if n is even).*

- *Commuting square of $Q_{\mathcal{G}}$ for each pair of non-related directions, i.e. $\delta_{A(K^c, i,j)} = \delta_{A(K^c, j,i)}$ if $a_{i,j} = 0$*

- $\delta_{A(K^c, i,i,j)} - (q + q^{-1})\delta_{A(K^c, i,j,i)} + \delta_{A(K^c, j,i,i)}$ *if $a_{i,j} = -1$.*

Proof: Translate using χ_q, the type **II** relations defining $\mathsf{u}_q^+(\mathcal{G})$ as a quotient of $\mathsf{v}_q^+(\mathcal{G})$ to $k^{Q_{\mathcal{G}}}$.

The interest of the above presentation of $\mathsf{u}_q^+(\mathcal{G})$ is that J is an *admissible* ideal of the path algebra $k^{Q_{\mathcal{G}}}$, which means that $F^m \subset J \subset F^2$ where F is the two-sided ideal of functions with support contained in the set of positive length paths and m is a positive integer. Indeed, observe that F^l is the two-sided ideal generated by the Dirac masses of paths of length l, hence each generator of J is contained in F^2. Conversely, the image of F in $k^{Q_{\mathcal{G}}}/J$ is nilpotent as a consequence of the Poincaré-Birkhof-Witt theorem for $\mathsf{u}_q^+(\mathcal{G})$, see 5.10 of [28].

An admissible quotient k^Q/J of a path algebra has a special feature, its Jacobson radical is F/J as this nilpotent ideal provides a semisimple algebra $\times_{u \in Q_0} k\delta_u$ at the quotient. Since simple modules for the later are one-dimensional – the complete list is provided by $\{k\delta_u\}_{u \in Q_0}$ – the same property holds for k^Q/J. Of course the radical F/J has zero action on $k\delta_u$ while δ_u acts as 1 and δ_v acts as 0 if $v \neq u$. Moreover it is no difficult to show that the dimension of $\mathrm{Ext}^1_{k^Q/J}(k\delta_u, k\delta_v)$ is the number of arrows from u to v. These well known facts specializes as follows:

Proposition 2.5 *The irreducible $\mathsf{u}_q^+(\mathcal{G})$-modules are one-dimensional and in one-to-one correspondence with $G = <K_1, \ldots, K_t \mid K_i K_j = K_j K_i, \ K_i^n = 1>$. Let S_{K^c} and S_{K^d} be simple modules corresponding to K^c and K^d. Then $\dim_k \mathrm{Ext}^1_{\mathsf{u}_q^+(\mathcal{G})}(S_{K^d}, S_{K^c}) = 1$ if $K^d = K^{c-a_{-,i}}$ for some i, and 0 otherwise.*

An associative algebra is *connected* if 0 and 1 are the only central idempotents. Otherwise the connected components (or blocks) of the algebra are the sub-algebras in its decomposition as a product of connected sub-algebras, given by the complete system of central primitive idempotents (such a system is unique). In case the algebra is an admissible quotient of a path algebra of a quiver, the connected components of the algebra corresponds to the connected components of the quiver.

Proposition 2.6 *The algebra* $\mathsf{u}_q^+(\mathcal{G})$ *is connected if and only if* $\det C$ *is invertible in* $\mathbf{Z}/n\mathbf{Z}$. *More precisely, the number of connected components of* $\mathsf{u}_q^+(\mathcal{G})$ *is the cardinality of* $\operatorname{coker} C$, *considering the Cartan matrix* C *as performing an endomorphism of* $(\mathbf{Z}/n\mathbf{Z})^t$.

Proof: If G is a group and S is a subset of G, the Cayley graph has set of vertices G and each vertex g is the target of an arrow coming from gs^{-1} for each $s \in S$. Clearly the number of connected components of such a graph is the index of the subgroup generated by S in G. In our context S is given by the columns of C as elements of $G = (\mathbf{Z}/n\mathbf{Z})^t$ and the result follows.

We end this section with an observation concerning the comultiplication of $\mathsf{u}_q^+(\mathcal{G})$, suggested by the presentation obtained above. In order to simplify the notation, we denote $\alpha(K^c,\ i)$ the Dirac mass pointing out an arrow $A(K^c,\ i)$. Hence the trivial paths gives a complete set of primitive orthogonal idempotents, $K^d \alpha(K^c,\ i) = 0$ if $K^d \neq K^c$ and $K^c \alpha(K^c,\ i) = \alpha(K^c,\ i)$. Moreover, $\alpha(K^c,\ i)\, K^{c-a_{-,i}} = \alpha(K^c,\ i)$ and $\alpha(K^c,\ i)\, K^d = 0$ if K^d is not the source vertex of $\alpha(K^c,\ i)$, that is if $K^d \neq K^{c-a_{-,i}}$.

We will consider a crossed tensor product algebra $\mathsf{u}_q^+(\mathcal{G}) \underline{\otimes} \mathsf{u}_q^+(\mathcal{G})$. First notice that the usual tensor product of algebras $\mathsf{u}_q^+(\mathcal{G}) \otimes \mathsf{u}_q^+(\mathcal{G})$ is also an admissible quotient of the path algebra of a suitable quiver. Indeed, let $Q \times Q$ be the quiver with set of vertices $Q_0 \times Q_0$ and set of arrows $Q_1 \times Q_0 \coprod Q_0 \times Q_1$ together with the obvious source and terminus maps. The tensor product of algebras $k^Q \otimes k^Q$ can be identified with the quotient of the path algebra $k^{Q \times Q}$ by the commuting relations provided by the mixed squares, i.e.

$$(t\alpha, \beta)(\alpha, s\beta) = (\alpha, t\beta)(s\alpha, \beta)$$

for every couple of arrows α and β. In order to present $k^Q/J \otimes k^Q/J$, we just need to consider $k^{Q \times Q}$ modulo the preceeding commuting relations together with the "propagation" of J, namely the two-sided ideal generated by (J, K^c) and (K^c, J) for each vertex K^c.

Instead of the commuting relations above, we consider the following relations provided by each mixed squares:

$$q^{a_{i,j}} \left(K^c, \ \alpha(K^d, \ j) \right) \left((\alpha(K^c, \ i), \ K^{d-a}\text{-},j) = \left(\alpha(K^c, \ i), \ K^d \right) \left(K^{c-a}\text{-},i, \ \alpha(K^d, j) \right)$$

as well as the unchanged quoted propagation of J. The admissible quotient of $k^{Q_{\mathcal{G}}} \otimes k^{Q_{\mathcal{G}}}$ by those relations presents a crossed tensor product denoted $\mathsf{u}_q^+(\mathcal{G}) \underline{\otimes} \mathsf{u}_q^+(\mathcal{G})$, which means that $\mathsf{u}_q^+(\mathcal{G}) \otimes 1$ and $1 \otimes \mathsf{u}_q^+(\mathcal{G})$ are sub-algebras identical to $\mathsf{u}_q^+(\mathcal{G})$ and $(x \otimes 1)(1 \otimes y) = (x \otimes y)$. However the reverse tensor product differs, it is provided by the consequences of the q-commuting mixed square relations above.

Proposition 2.7 *There exist an algebra automorphism ϕ of the crossed product $\mathsf{u}_q^+(\mathcal{G}) \underline{\otimes} \mathsf{u}_q^+(\mathcal{G})$ given by $\phi(K^x, \ K^y) = (K^y, \ K^x)$ in degree zero, and by*

$$\phi\left(K^x, \ \alpha(K^y, \ i) \right) = \ q^{-x_i} \ \left(\alpha(K^y, \ i), \ K^x \right)$$

$$\phi\left(\alpha(K^x, \ i), \ K^y \right) = \ q^{y_i} \ \left(K^y, \ \alpha(K^x, \ i) \right)$$

in degree one, verifying $\phi\Delta = \Delta^{\mathrm{op}}$ where Δ is the comultiplication of $\mathsf{u}_q^+(\mathcal{G})$.

3 Representation type

We will study in this Section the representation type of $\mathsf{u}_q^+(\mathcal{G})$ using two methods which can be useful in order to produce families of indecomposable modules. Actually the first method do not reach the case $t = 3$, i.e. $\mathcal{G} = sl_3$ or $\mathcal{G} = sl_2 \times sl_2$.

Proposition 3.1 *The algebra $\mathsf{u}_q^+(\mathcal{G})$ is of wild representation type for $n \geq 5$ and $t \geq 3$.*

Proof: We consider the 2-nilpotent quotient of $\mathsf{u}_q^+(\mathcal{G})$, namely the path algebra $k^{Q_{\mathcal{G}}}$ modulo F^2, which is an algebra having zero square Jacobson radical. The classification of algebras k^Q/F^2 according to their representation type is known: first consider the separated quiver Q' which has double set of vertices $Q_0 \times \{0, 1\}$ and same set of arrows Q_1, but source and terminus maps given by $s'(a) = (s(a), 0)$ and $t'(a) = (t(a), 1)$. It is easy to see that except for simple modules, indecomposable k^Q/F^2 and $k^{Q'}$-modules corresponds naturally (the second algebra has twice the number of simple modules of the first one).

Next the representation type of the path algebra $k^{Q'}$ is determined by the underlying graph of Q' (see [1, 24]). If this graph is of Dynkin type or extended Dynkin (affine) type, the representation type is respectively finite or tame. Otherwise the representation type is wild. For the Cayley graph $Q_{\mathcal{G}}$ the separated quiver $Q'_{\mathcal{G}}$ has at least 3 arrows leaving each zero level vertex, which insures that $k^{Q'_{\mathcal{G}}}$ is of wild representation type.

The second method is based on the usual presentation of $\mathsf{u}_q^+(\mathcal{G})$, we will show that the sub-algebra $\mathsf{u}_q^{++}(\mathcal{G})$ generated by $E_1, \ldots, E_t$ is already wild, except for sl_2. This is not enough to obtain the same property for the entire algebra: consider the trivial fact that any finite dimensional algebra is a sub-algebra of its linear endomorphisms algebra which has only one indecomposable – actually simple – module. We will see that there exist a $\mathsf{u}_q^{++}(\mathcal{G})$- sub-bimodule of $\mathsf{u}_q^+(\mathcal{G})$ complementing $\mathsf{u}_q^{++}(\mathcal{G})$. The next result is due to V.M. Bondarenko and Y.A. Drozd, see [3] and [27]. It asserts that when a complement as before exists, the sub-algebra provides a lower bound for the representation type of the entire algebra.

Proposition 3.2 *Let Λ be a finite dimensional algebra and let A be a sub-algebra. Assume there exist an A sub-bimodule B of the A-bimodule Λ such that $\Lambda = A \oplus B$. Then if the representation type of A is wild, the representation type of Λ is also wild.*

Proposition 3.3 *The algebra $\mathsf{u}_q^+(\mathcal{G})$ is of wild representation type for $n \geq 5$ and $t \geq 2$ and of finite representation type for $t = 1$, that is for $\mathcal{G} = sl_2$.*

Proof: Consider the quasipolynomial quotient algebra of $\mathsf{u}_q^{++}(\mathcal{G})$ given by generators $B_1, \ldots, B_t$ subjected to $B_i^e = 0$, $B_i B_j = B_j B_i$ if $a_{i,j} = 0$ and $B_i B_j = q B_j B_i$ if $a_{i,j} = -1$ and $i > j$. Since $t \geq 2$ and $n \geq 5$, we consider further the quotient algebra generated by C_1 and C_2, subjected to $C_1^3 = C_2^3 = 0$ and $C_2 C_1 = C_1 C_2$ if $a_{1,2} = 0$ or $C_2 C_1 = q C_1 C_2$ if $a_{1,2} = -1$. The later has a known wild quotient, namely the algebra (c), p.238 of [36] on generators X, Y and relations $X^2 = 0$, $XY - \alpha Y X = 0$, $Y^2 X = Y^3 = 0$ whith $\alpha \neq 0$. The case of sl_2 and q any root of unity has been studied in [5], notice however that $\mathsf{u}_q^{++}(sl_2) = k[E]/E^n$ is already of finite representation type.

In order to describe a $\mathsf{u}_q^{++}(\mathcal{G})$ sub-bimodule of $\mathsf{u}_q^+(\mathcal{G})$ complementing $\mathsf{u}_q^{++}(\mathcal{G})$, we recall the existence of a Poincaré-Birkhof-Witt basis. The root

vectors obtained through the action of the braid group corresponding to the Cartan matrix gives a basis of monomials $K_1^{m_1} \ldots K_t^{m_t} E^s$, see for instance [7]. A vectorial complement is provided by the linear span of elements with $m \neq 0$. As a consequence of the commutation relations between E_i and K_j, the later vector space is indeed a $\mathsf{u}_q^{++}(\mathcal{G})$ sub-bimodule.

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